

Strong Predictor-Corrector Euler-Maruyama Methods for Stochastic Differential Equations with Markovian Switching

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Abstract

In this paper numerical methods for solving stochastic differential equations with Markovian switching (SDEwMSs) are developed by pathwise approximation. The proposed family of strong predictor-corrector Euler-Maruyama methods is designed to overcome the propagation of errors during the simulation of an approximate path. This paper not only shows the strong convergence of the numerical solution to the exact solution but also reveals the order of the error under some conditions on the coefficient functions. A natural analogue of p -stability criterion is studied. Numerical examples are given to illustrate the computational efficiency of the new predictor-corrector Euler-Maruyama approximation.

Keywords: Strong Predictor-Corrector Euler-Maruyama methods, Markovian switching, Numerical solutions

1 Introduction

Stochastic differential equations with Markovian switching (SDEwMSs) arise in mathematics models of hybrid systems that possess frequent unpredictable structural changes. One of the distinct features of such systems is that the underlying dynamics are subject to changes with respect to certain configurations. Such models have been used with great success in a variety of application areas, including flexible manufacturing systems, electric power networks, risk theory, financial engineering and insurance modeling, we refer the readers to Arapostathis, Ghosh and Marcus [1], Jobert and Rogers [7], Mao and Yuan [11], Rolski, Schmidli, Schmidt and Teugels [14], Smith [15], Yang and Yin [16] and references therein.

Generally, although the fundamental theories such as existence and uniqueness of the solution as well as stability of SDEwMSs have been well studied, most of SDEwMSs cannot be solved analytically. Thus, appropriate numerical approximation methods such as the Euler (or Euler-Maruyama) method are needed to apply SDEwMSs in practice or to study their properties.

Yuan and Mao [18] firstly considered the numerical solutions of the following stochastic differential equations with Markovian switching

$$dy(t) = f(y(t), r(t))dt + g(y(t), r(t))dW(t), \quad (1.1)$$

here $y(t)$ is referred to the state while $r(t)$ is regarded as the mode. The system will switch from one mode to another in a random way, and the switching between the modes is governed by a Markov chain. They proved the mean-square convergence of the Euler-Maruyama(EM) approximation for this hybrid stochastic systems, and the order of error was also estimated. Yin, Song and Zhang [17] extended (1.1) to a family of more general jump-diffusions with Markovian switching, and proved the numerical solutions based on finite-difference procedure weak converge to the desired limit by means of a martingale problem formulation.

During recent years, there also exist extensive literatures which prove the convergence of the Euler-Maruyama method applied to some stochastic differential equation with some additional feature, like including some sort of delay, jumps, Markovian switching or combinations thereof, see for example, Bruti-Liberati and Platen [2], Hou, Tong and Zhang [6], Li and Hou [9], Mao and Yuan [11], Rathinasamy and Balachandran [13], among others. The corresponding proof is basically the same each time, the only novelty coming from changing it a bit to deal with the additional feature.

It is well known that Euler-Maruyama method and most other explicit schemes for solving stochastic differential equations (SDEs) work unreliably and sometimes generate large errors, see for instance Milstein, Platen and Schurz [10], implicit and predictor-corrector schemes are designed to achieve improved numerical stability and turn out to be better suited to simulation task. Generally, implicit schemes usually cost significant computational time and are sometimes not reliably accomplished, however, this phenomenon can be avoided when using some appropriate discrete time schemes, including predictor-corrector methods. In Kloeden and Platen [8], predictor-corrector methods have been proposed as weak discrete time approximations for solving SDEs, which can be used in Monte Carlo simulation. For the strong discrete time approximation of solutions of SDEs, a family of predictor-corrector Euler methods has been developed in Bruti-Liberati and Platen [3]. However, there are no strong predictor-corrector methods available for SDEwMSs yet.

In this paper, we develop a new family of strong predictor-corrector Euler-Maruyama (PCEM) methods for SDEwMSs (1.1), which are shown to converge with strong order 0.5, and demonstrate their performance by considering some examples.

The rest of the paper is arranged as follows. In Section 2 we introduce some necessary notations and define a family of strong predictor-corrector Euler-Maruyama approximate solutions to SDEwMSs. In Section 3 we show

that the PCEM solutions converge to the exact solution in L^2 under the global Lipschitz condition and reveal the order of convergence is 0.5. In Section 4 we extend the PCEM convergence results to multi-dimensional case under certain conditions. In Section 5 the numerical stability of SDEwMSs will be introduced and discussed. Finally, in Section 6 some numerical examples are given and compared for simulated paths with different degrees of implicitness to illustrate the computational efficiency of the predictor-corrector Euler-Maruyama approximation.

2 Preliminary and algorithm

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions. Suppose that there is a finite set $S = \{1, 2, \dots, N\}$, representing the possible regimes of the environment. We work with a finite-time horizon $[0, T]$ for some $T > 0$. Let $f(\cdot, \cdot) : \mathbb{R}^d \times S \rightarrow \mathbb{R}^d$, $g(\cdot, \cdot) : \mathbb{R}^d \times S \rightarrow \mathbb{R}^{d \times d}$ be both Borel measurable. Consider the dynamic system given by (1.1) with initial value $y(0) = y_0 \in \mathbb{R}^d$ and $r(0) = i_0 \in S$, where $W(t) = (W^1(t), \dots, W^d(t))^T$ is a d -dimensional \mathcal{F}_t -adapted standard Brownian motion, and $r(t)$ is a continuous-time Markov chain taking value in a finite state space $S = \{1, 2, \dots, N\}$ with the generator $Q = (q_{ij})_{N \times N}$ given by

$$P\{r(t + \delta) = j | r(t) = i\} = \begin{cases} q_{ij}\delta + o(\delta), & \text{if } i \neq j, \\ 1 + q_{ii}\delta + o(\delta), & \text{if } i = j, \end{cases}$$

provided $\delta \downarrow 0$, and

$$-q_{ii} = \sum_{i \neq j} q_{ij} < +\infty.$$

We assume $W(t)$ and $r(t)$ are independent. Throughout this paper, we denote by $|\cdot|$ the Euclidean norm for vectors or the trace norm for matrices.

2.1 Existence and uniqueness

Under certain conditions we can establish the existence of a pathwise unique solution of (1.1). Here we make the following global Lipschitz (GL) and linear growth (LG) assumptions:

(H1) GL: For all $(x, i), (y, i) \in \mathbb{R}^d \times S$, there exists a constant $L_1 > 0$ such that

$$|f(x, i) - f(y, i)|^2 + |g(x, i) - g(y, i)|^2 \leq L_1 |x - y|^2.$$

(H2) LG: For all $(x, i) \in \mathbb{R}^d \times S$, there exists a constant $L_2 > 0$ such that

$$|f(x, i)|^2 \vee |g(x, i)|^2 \leq L_2(1 + |x|^2).$$

Remarks 2.1. It is easy to show that if $f(\cdot, \cdot)$, $g(\cdot, \cdot)$ satisfy GL condition, then they also satisfy LG condition, but for the convenience of the reader we preserve it.

The following theorem guarantee the existence and uniqueness of the solution to equation (1.1), which are of use in studying some numerical schemes.

Theorem 2.1. *If $f(x, i)$, $g(x, i)$ satisfy the conditions $(\mathcal{H}1)$, $(\mathcal{H}2)$, and suppose $W(t), r(t)$ be independent. Then there exists a unique d -dimensional \mathcal{F}_t -adapted right-continuous process $y(t)$ with left-hand limits which satisfies equation (1.1) such that $y(0) = y_0$ and $r(0) = i_0$ a.s.*

Proof. See Theorem 3.13 in Mao and Yuan [11]. \square

2.2 Algorithm

Now we turn our attention to numerical algorithm. For convenience, we first consider one-dimensional SDEwMSs. Given $\Delta > 0$ as a step size, denote $\{t_i\}_{i \geq 1}$ the usual equidistant time discretization of a bounded interval $[0, T]$, i.e. $t_0 = 0$, $t_i - t_{i-1} = \Delta$, if $t_{n-1} < T \leq t_n$ then set $t_n = T$. Denote $\Delta W_{t_k} = W(t_{k+1}) - W(t_k)$.

For given partition $\{t_k\}_{k \geq 1}$, $\{r(t_k)\}_{k \geq 1}$ is a discrete Markov chain with transition probability matrix $(P(i, j))_{N \times N}$, here $P(i, j) = P(r(t_{k+1}) = j | r(t_k) = i)$ is the ij th entry of the matrix $e^{(t_{k+1}-t_k)Q}$, thus we could use following recursion procedure to simulate the discrete Markov chain $\{r(t_k)\}_{k \geq 1}$, suppose $r(t_k) = i_1$ and generate a random number ξ which is uniformly distributed in $[0, 1]$, then we define

$$r(t_{k+1}) = \begin{cases} i_2, & \text{if } i_2 \in S - \{N\} \text{ and } \sum_{j=1}^{i_2-1} P(i_1, j) \leq \xi < \sum_{j=1}^{i_2} P(i_1, j), \\ N, & \text{if } \sum_{j=1}^{N-1} P(i_1, j) \leq \xi. \end{cases}$$

Repeating this procedure a trajectory of $\{r(t_k)\}_{k \geq 1}$ can be simulated.

Now we can introduce a new family of strong predictor-corrector Euler-Maruyama(PCEM) methods for SDEwMSs. Given initial value $Y_{t_0} = y_0 \in \mathbb{R}$ and $r_{t_0} = i_0 \in S$, the proposed family of strong PCEM is given by the predictor

$$\tilde{Y}_{t_{k+1}} = Y_{t_k} + f(Y_{t_k}, r_{t_k})\Delta + g(Y_{t_k}, r_{t_k})\Delta W_{t_k}, \quad (2.1)$$

and by the corrector

$$\begin{aligned} Y_{t_{k+1}} = & Y_{t_k} + \{\theta \bar{f}_\eta(\tilde{Y}_{t_{k+1}}, r_{t_k}) + (1 - \theta) \bar{f}_\eta(Y_{t_k}, r_{t_k})\} \Delta \\ & + \{\eta g(\tilde{Y}_{t_{k+1}}, r_{t_k}) + (1 - \eta) g(Y_{t_k}, r_{t_k})\} \Delta W_{t_k}. \end{aligned} \quad (2.2)$$

Here parameters $\theta, \eta \in [0, 1]$ denote the degree of implicitness in the drift and the diffusion coefficients, respectively, and $\bar{f}_\eta(x, i)$ is defined as

$$\bar{f}_\eta(x, i) = f(x, i) - \eta g(x, i) \frac{\partial g(x, i)}{\partial x}, \eta \in [0, 1], \quad (2.3)$$

which is called the corrected drift function. This scheme can be written in the form

$$Y_{t_{k+1}} = Y_{t_k} + f(Y_{t_k}, r_{t_k})\Delta + g(Y_{t_k}, r_{t_k})\Delta W_{t_k} + \sum_{l=1}^4 R_{l,k}, \quad (2.4)$$

where

$$R_{1,k} = \theta \{f(\tilde{Y}_{t_{k+1}}, r_{t_k}) - f(Y_{t_k}, r_{t_k})\}\Delta, \quad (2.5)$$

$$R_{2,k} = -\theta \eta \{g(\tilde{Y}_{t_{k+1}}, r_{t_k}) \frac{\partial g(\tilde{Y}_{t_{k+1}}, r_{t_k})}{\partial x} - g(Y_{t_k}, r_{t_k}) \frac{\partial g(Y_{t_k}, r_{t_k})}{\partial x}\}\Delta, \quad (2.6)$$

$$R_{3,k} = -\eta g(Y_{t_k}, r_{t_k}) \frac{\partial g(Y_{t_k}, r_{t_k})}{\partial x} \Delta, \quad (2.7)$$

$$R_{4,k} = \eta \{g(\tilde{Y}_{t_{k+1}}, r_{t_k}) - g(Y_{t_k}, r_{t_k})\}\Delta W_{t_k}. \quad (2.8)$$

For each $t \in [t_k, t_{k+1})$, let

$$\bar{Y}(t) = Y_{t_k}, \bar{r}(t) = r_{t_k}, \bar{R}_l(t) = \frac{R_{l,k}}{\Delta}, l = 1, 2, 3, \bar{R}_4(t) = \frac{R_{4,k}}{\Delta W_{t_k}}. \quad (2.9)$$

Therefore, we can define the continuous approximation solution $Y(t)$ on the entire interval $[0, T]$ by

$$\begin{aligned} Y(t) = y_0 &+ \int_0^t f(\bar{Y}(s), \bar{r}(s))ds + \int_0^t g(\bar{Y}(s), \bar{r}(s))dW(s) \\ &+ \sum_{l=1}^3 \int_0^t \bar{R}_l(s)ds + \int_0^t \bar{R}_4(s)dW(s). \end{aligned} \quad (2.10)$$

Note that $Y(t_k) = \bar{Y}(t_k) = Y_{t_k}$, which means $Y(t)$ and $\bar{Y}(t)$ coincide with the discrete approximate solution at the gridpoints.

Remarks 2.2. The major advantage of the above PCEM approximate schemes is that there are flexible degrees of implicitness parameters θ and η to choose for simulating paths properly. For the case $\theta = \eta = 0$ one recovers the Euler-Maruyama scheme which is well discussed in Yuan and Mao [18].

3 Convergence with the global Lipschitz(GL) and linear growth(LG) conditions

In this section, we will prove that the numerical solution $Y(t)$ converges to the exact solution $y(t)$ in L^2 as step size $\Delta \downarrow 0$, and the order of convergence is one-half. To begin with, we need the following GL condition and LG condition for $\bar{f}_\eta(\cdot, \cdot)$.

(H3) GL: For all $(x, i), (y, i) \in \mathbb{R} \times S$, there exists a constant $L_3 > 0$ such that

$$|\bar{f}_\eta(x, i) - \bar{f}_\eta(y, i)|^2 \leq L_3 |x - y|^2.$$

(H4) LG: For all $(x, i) \in \mathbb{R} \times S$, there exists a constant $L_4 > 0$ such that

$$|\bar{f}_\eta(x, i)|^2 \leq L_4(1 + |x|^2).$$

We are now ready to present the key results of this section which are stated as following.

Theorem 3.1. *Assume the SDEwMSs (1.1) defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ satisfying*

a) $W(t), r(t)$ are independent,

b) $f(\cdot, \cdot), g(\cdot, \cdot), \bar{f}_\eta(\cdot, \cdot)$ satisfy conditions (H1), (H2), (H3) and (H4),

then the unique strong solution $y(t)$ and numerical solution $Y(t)$ obtained in section 2.2 satisfying:

$$E\left(\sup_{0 \leq t \leq T} |Y(t) - y(t)|^2\right) \leq C\Delta + o(\Delta), \quad (3.1)$$

where C is a positive constant independent of Δ .

In order to give the proof of this theorem, we first provide a number of useful lemmas. The first two lemmas show that the continuous approximation has bounded moments in a strong sense. The latter lemmas play an important role in proving the strong convergence result, which mainly refer to Bruti-Liberati and Platen [3].

Lemma 3.1. *Under conditions (H1), (H2), (H3) and (H4), for any $p \geq 2$, there exists a constant M which is dependent on p, T, L and y_0 , but independent of Δ , such that*

$$E\left(\sup_{0 \leq t \leq T} |Y(t)|^p\right) \leq M. \quad (3.2)$$

We omit the proof since it is similar to Lemma 4.1 in Mao and Yuan [11].

Lemma 3.2. *Under conditions $(\mathcal{H}1)$, $(\mathcal{H}2)$, $(\mathcal{H}3)$ and $(\mathcal{H}4)$, there exists a constant M which is dependent on T , L and y_0 , but independent of Δ , such that*

$$E(\max_{0 \leq k \leq n_T} |Y_{t_k}|^2) \leq M. \quad (3.3)$$

This is an immediate result of Lemma 3.1, since $Y(t_k) = \bar{Y}(t_k) = Y_{t_k}$.

Lemma 3.3. *There exists a constant C which is dependent on T , L and y_0 , but independent of Δ , such that*

$$\sum_{l=1}^4 E[\max_{1 \leq n \leq n_T} |\sum_{0 \leq k \leq n-1} R_{l,k}|^2] \leq C\Delta. \quad (3.4)$$

Proof. By the Cauchy-Schwarz inequality and the GL condition, from equation (2.5), we can obtain

$$\begin{aligned} & E[\max_{1 \leq n \leq n_T} |\sum_{0 \leq k \leq n-1} R_{1,k}|^2] \\ &= E[\max_{1 \leq n \leq n_T} |\sum_{0 \leq k \leq n-1} \theta\{f(\tilde{Y}_{t_{k+1}}, r_{t_k}) - f(Y_{t_k}, r_{t_k})\}\Delta|^2] \\ &\leq E[\max_{1 \leq n \leq n_T} (\sum_{0 \leq k \leq n-1} |\theta\Delta|^2)(\sum_{0 \leq k \leq n-1} |f(\tilde{Y}_{t_{k+1}}, r_{t_k}) - f(Y_{t_k}, r_{t_k})|^2)] \\ &\leq C\Delta E[(\sum_{0 \leq k \leq n_T-1} \Delta)(\sum_{0 \leq k \leq n_T-1} |f(\tilde{Y}_{t_{k+1}}, r_{t_k}) - f(Y_{t_k}, r_{t_k})|^2)] \\ &\leq C\Delta E[\sum_{0 \leq k \leq n_T-1} |\tilde{Y}_{t_{k+1}} - Y_{t_k}|^2] \\ &\leq C\Delta E[\sum_{0 \leq k \leq n_T-1} (E[|f(Y_{t_k}, r_{t_k})\Delta|^2|\mathcal{F}_{t_k}] + E[|g(Y_{t_k}, r_{t_k})\Delta W_{t_k}|^2|\mathcal{F}_{t_k}])]. \end{aligned} \quad (3.5)$$

Then by using the Cauchy-Schwarz inequality, the Itô's isometry, the LG condition and Lemma 3.2, we get

$$\begin{aligned} & E[\max_{1 \leq n \leq n_T} |\sum_{0 \leq k \leq n-1} R_{1,k}|^2] \\ &\leq C\Delta E[\int_0^{t_{n_T}} E((1 + |Y_{t_{n_z}}|^2)|\mathcal{F}_{t_{n_z}})dz] \\ &\leq C\Delta \int_0^{t_{n_T}} E(1 + \max_{0 \leq n \leq n_T} |Y_{t_n}|^2)dz \\ &\leq C\Delta. \end{aligned} \quad (3.6)$$

With the similar steps as in (3.5) and (3.6), we have

$$E[\max_{1 \leq n \leq n_T} |\sum_{0 \leq k \leq n-1} R_{2,k}|^2] \leq C\Delta. \quad (3.7)$$

It is also easy to show by the LG condition and Lemma 3.2 that

$$E[\max_{1 \leq n \leq n_T} |\sum_{0 \leq k \leq n-1} R_{3,k}|^2] \leq C\Delta. \quad (3.8)$$

By using Doob's martingale inequality, the Itô's isometry, the GL condition and equation (2.8), we have

$$\begin{aligned} & E[\max_{1 \leq n \leq n_T} |\sum_{0 \leq k \leq n-1} R_{4,k}|^2] \\ &= E[\max_{1 \leq n \leq n_T} |\sum_{0 \leq k \leq n-1} \eta \int_{t_k}^{t_{k+1}} (g(\tilde{Y}_{t_{k+1}}, r_{t_k}) - g(Y_{t_k}, r_{t_k})) dW(z)|^2] \\ &\leq CE[|\int_0^T (g(\tilde{Y}_{t_{n_z+1}}, r_{t_{n_z}}) - g(Y_{t_{n_z}}, r_{t_{n_z}})) dW(z)|^2] \\ &= C \int_0^T E[|g(\tilde{Y}_{t_{n_z+1}}, r_{t_{n_z}}) - g(Y_{t_{n_z}}, r_{t_{n_z}})|^2] dz \\ &\leq C \int_0^T E[|\tilde{Y}_{t_{n_z+1}} - Y_{t_{n_z}}|^2] dz. \end{aligned} \quad (3.9)$$

Therefore, with similar steps as in (3.5) and (3.6), we also have

$$\begin{aligned} & E[\max_{1 \leq n \leq n_T} |\sum_{0 \leq k \leq n-1} R_{4,k}|^2] \\ &\leq C \int_0^T E[\int_{t_{n_z}}^{t_{n_z+1}} E[(1 + |Y_{t_{n_z}}|^2) |\mathcal{F}_{t_{n_z}}] ds] dz \\ &\leq C \int_0^T E[\int_{t_{n_z}}^{t_{n_z+1}} E[(1 + \max_{0 \leq n \leq n_T} |Y_{t_n}|^2) |\mathcal{F}_{t_{n_z}}] ds] dz \\ &\leq C\Delta. \end{aligned} \quad (3.10)$$

Thus the required assertion follows. \square

Lemma 3.4. *Under conditions $(\mathcal{H}1)$, $(\mathcal{H}2)$, $(\mathcal{H}3)$ and $(\mathcal{H}4)$, then for any $t \in [t_k, t_{k+1})$, we have*

$$\sum_{l=1}^3 E[\sup_{t_k \leq s \leq t} |\int_{t_k}^s \frac{R_{l,k}}{\Delta} dz|^2] + E[\sup_{t_k \leq s \leq t} |\int_{t_k}^s \frac{R_{4,k}}{\Delta} dW(z)|^2] \leq o(\Delta). \quad (3.11)$$

The proof of this lemma is similar to that in Lemma 3.3.

Now we are in a position to prove our Theorem 3.1.

Proof of Theorem 3.1 : From equation (2.10), we have

$$\begin{aligned}
Z(t) &= E\left(\sup_{0 \leq s \leq t} |Y(s) - y(s)|^2\right) \\
&= E\left[\sup_{0 \leq s \leq t} \left| \int_0^s (f(\bar{Y}(z), \bar{r}(z)) - f(y(z), r(z))) dz \right. \right. \\
&\quad \left. \left. + \int_0^s (g(\bar{Y}(z), \bar{r}(z)) - g(y(z), r(z))) dW(z) \right. \right. \\
&\quad \left. \left. + \sum_{l=1}^3 \int_0^s \bar{R}_l(z) dz + \int_0^s \bar{R}_4(z) dW(z) \right|^2\right]. \tag{3.12}
\end{aligned}$$

Let $n = \lfloor t/\Delta \rfloor$, the integer part of t/Δ . Then, by Hölder inequality, Doob's martingale inequality and equation (2.9), we have

$$\begin{aligned}
Z(t) &= E\left(\sup_{0 \leq s \leq t} |Y(s) - y(s)|^2\right) \\
&\leq CE \int_0^t |f(\bar{Y}(z), \bar{r}(z)) - f(y(z), r(z))|^2 dz \\
&\quad + CE \int_0^t |g(\bar{Y}(z), \bar{r}(z)) - g(y(z), r(z))|^2 dz \\
&\quad + C \sum_{l=1}^4 E\left[\max_{1 \leq m \leq n} \left| \sum_{0 \leq k \leq m-1} R_{l,k} \right|^2\right] + C \sum_{l=1}^3 E\left[\sup_{t_k \leq s \leq t} \left| \int_{t_k}^s \frac{R_{l,k}}{\Delta} dz \right|^2\right] \\
&\quad + CE\left[\sup_{t_k \leq s \leq t} \left| \int_{t_k}^s \frac{R_{4,k}}{\Delta} dW(z) \right|^2\right]. \tag{3.13}
\end{aligned}$$

We focus on the last three parts of the right side. From Lemma 3.3 and Lemma 3.4, we have

$$\begin{aligned}
&C \sum_{l=1}^4 E\left[\max_{1 \leq m \leq n} \left| \sum_{0 \leq k \leq m-1} R_{l,k} \right|^2\right] + C \sum_{l=1}^3 E\left[\sup_{t_k \leq s \leq t} \left| \int_{t_k}^s \frac{R_{l,k}}{\Delta} dz \right|^2\right] \\
&+ CE\left[\sup_{t_k \leq s \leq t} \left| \int_{t_k}^s \frac{R_{4,k}}{\Delta} dW(z) \right|^2\right] \leq C\Delta + o(\Delta). \tag{3.14}
\end{aligned}$$

Let I_G be the indicator function for set G . Let $t \in [t_k, t_{k+1})$. From (2.1)–(2.10), obviously, we have $\bar{Y}(t_k)$ and $I_{\{r(t) \neq r(t_k)\}}$ are conditionally independent with respect to the σ -algebra generated by $r(t_k)$. So by the same procedures as in Theorem 3.1 in Yuan and Mao [18], we can obtain

$$\begin{aligned}
&E \int_0^t |f(\bar{Y}(s), \bar{r}(s)) - f(y(s), r(s))|^2 ds \\
&\leq 2L^2 \int_0^t E|\bar{Y}(s) - y(s)|^2 ds + C\Delta + o(\Delta), \tag{3.15}
\end{aligned}$$

$$\begin{aligned}
& E \int_0^t |b(\bar{Y}(s), \bar{r}(s)) - b(y(s), r(s))|^2 ds \\
& \leq 2L^2 \int_0^t E |\bar{Y}(s) - y(s)|^2 ds + C\Delta + o(\Delta).
\end{aligned} \tag{3.16}$$

Substituting (3.14), (3.15), (3.16) into (3.13) shows that

$$E\left(\sup_{0 \leq s \leq t} |Y(s) - y(s)|^2\right) \leq C \int_0^t E |\bar{Y}(s) - y(s)|^2 ds + C\Delta + o(\Delta). \tag{3.17}$$

Note that

$$E|\bar{Y}(s) - y(s)|^2 \leq 2E|Y(s) - y(s)|^2 + 2E|\bar{Y}(s) - Y(s)|^2. \tag{3.18}$$

Suppose $t_k \leq s < t_{k+1}$, by (2.10), Lemma 3.3 and Lemma 3.4, we can then show in the same way as in the case of SDEs that

$$E|Y(s) - \bar{Y}(s)|^2 \leq C\Delta. \tag{3.19}$$

Substituting (3.18), (3.19), into (3.17) immediately shows that

$$\begin{aligned}
& E\left(\sup_{0 \leq s \leq t} |Y(s) - y(s)|^2\right) \\
& \leq C \int_0^t (E|Y(s) - y(s)|^2 + E|\bar{Y}(s) - Y(s)|^2) ds + C\Delta + o(\Delta) \\
& \leq C \int_0^t E\left(\sup_{0 \leq s \leq t} |Y(s) - y(s)|^2\right) ds + C\Delta + o(\Delta).
\end{aligned} \tag{3.20}$$

Therefore, from Gronwall inequality we obtain that

$$E\left(\sup_{0 \leq t \leq T} |Y(t) - y(t)|^2\right) \leq C\Delta + o(\Delta).$$

The proof is complete.

4 The general multi-dimensional case

The results derived in Section 3 can be easily generalized to the multi-dimensional case, we just summarize the related numerical schemes and the convergence results in this section.

Consider the solution $y(t) = \{(y^1(t), \dots, y^d(t))^T, t \geq 0\}$ of the d -dimensional SDEwMSs

$$y(t) = y(0) + \int_0^t f(y(s), r(s)) ds + \sum_{j=1}^m \int_0^t g^j(y(s), r(s)) dW^j(s),$$

for $t \geq 0$. Here $y(0) \in \mathbb{R}^d$ denotes the deterministic initial value, $W^j = \{W^j(t), t \geq 0\}$, $j \in \{1, 2, \dots, m\}$ is a standard Brownian motion. $r(t)$ is a

Markov chain. The function $f : \mathbb{R}^d \times S \mapsto \mathbb{R}^d$ has the k th component $f^k(\cdot, \cdot)$. The function $g^j : \mathbb{R}^d \times S \mapsto \mathbb{R}^d, j \in 1, 2, \dots, m$ has the k th component $g^{k,j}(\cdot, \cdot)$. We define the function \bar{f}_η for $\eta \in [0, 1]$ with the k th component

$$\bar{f}_\eta^k(x, i) = f^k(x, i) - \eta_k \sum_{j_1, j_2=1}^m \sum_{i=1}^d g^{i, j_1}(x, i) \frac{\partial g^{i, j_2}(x, i)}{\partial x^i}.$$

for $(x, i) \in \mathbb{R}^d \times S$, which satisfies the following GL condition and LG condition

(H3') GL: For all $(x, i), (y, i) \in \mathbb{R}^d \times S$, there exists a constant $L_3 > 0$ such that

$$|\bar{f}_\eta(x, i) - \bar{f}_\eta(y, i)|^2 \leq L_3 |x - y|^2.$$

(H4') LG: For all $(x, i) \in \mathbb{R}^d \times S$, there exists a constant $L_4 > 0$ such that

$$|\bar{f}_\eta(x, i)|^2 \leq L_4(1 + |x|^2).$$

The k th component of the proposed family of strong PCEM schemes is given by the predictor

$$\tilde{Y}_{t_{n+1}}^k = Y_{t_n}^k + f^k(Y_{t_n}, r_{t_n})\Delta + \sum_{j=1}^m g^{k,j}(Y_{t_n}, r_{t_n})\Delta W_{t_n}^j,$$

and by the corrector

$$\begin{aligned} Y_{t_{n+1}}^k = & Y_{t_n}^k + \{\theta_k \bar{f}_\eta^k(\tilde{Y}_{t_{n+1}}, r_{t_n}) + (1 - \theta_k) \bar{f}_\eta^k(Y_{t_n}, r_{t_n})\}\Delta \\ & + \sum_{j=1}^m \{\eta_k g^{k,j}(\tilde{Y}_{t_{n+1}}, r_{t_n}) + (1 - \eta_k) g^{k,j}(Y_{t_n}, r_{t_n})\}\Delta W_{t_n}^j, \end{aligned}$$

for $\theta_k, \eta_k \in [0, 1], k \in \{1, 2, \dots, d\}$.

Hence we can define the approximation solution $Y(t)$ similarly to equation (2.10), then we can derive the following theorem analogically.

Theorem 4.1. *Assume the SDEwMSs (1.1) defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ satisfying*

a) $W(t), r(t)$ are independent,

b) $f(\cdot, \cdot), g(\cdot, \cdot), \bar{f}_\eta(\cdot, \cdot)$ satisfy conditions (H1), (H2), (H3') and (H4'),

then the unique strong solution $y(t)$ and numerical solution $Y(t)$ satisfying:

$$E\left(\sup_{0 \leq t \leq T} |Y(t) - y(t)|^2\right) \leq C\Delta + o(\Delta),$$

where C is a positive constant independent of Δ .

5 Numerical Stability

In this section we consider numerical stability issues, extending the analysis in Platen and Shi [12] to the Markovian switching case. When simulating discrete time approximations of solutions of SDEwMSs, numerical stability is clearly as important as numerical efficiency. There have been various efforts made in the literature trying to study numerical stability for a given scheme approximating solutions of SDEs, see, for instance, Hofmann and Platen [5], Higham [4], Bruti-Liberati and Platen [3] and Platen and Shi [12]. Generally, for analyzing numerical stability, some specifically designed test equations are necessary to be introduced, the test SDEs used in the above literatures are linear SDEs with multiplicative noise defined as

$$dX_t = (1 - \frac{3}{2}\alpha)\lambda X_t dt + \sqrt{\alpha|\lambda|}X_t dW_t, \quad (5.1)$$

for every $t \geq 0$, where $X_0 \geq 0$, $\lambda < 0$ and $\alpha \in [0, 1)$. Its explicit solution is of the form

$$X_t = X_0 \exp\{(1 - \alpha)\lambda t + \sqrt{\alpha|\lambda|}W_t\}, t \geq 0 \quad (5.2)$$

As a unified criterion, Platen and Shi [12] proposed the concept of p -stability criterion, which means that a process is p -stable if in the long run its p th moment vanishes. Hence, for $p > 0$, $\lambda < 0$, p -stable in equation (5.1) may be characterized by

$$\lim_{t \rightarrow \infty} E(|X_t|^p) = 0 \quad \text{iff} \quad 0 \leq \alpha < \frac{1}{1 + \frac{p}{2}}.$$

Since for different combinations of values of $\lambda\Delta, \alpha$ and p with given time step size Δ , a discrete time approximation Y_t and the original continuous process X_t have different stability properties. To explore these differences, the concept of stability region is introduced. The stability region, denoted by Γ , is determined by those triplets $(\lambda\Delta, \alpha, p) \in (-\infty, 0) \times [0, 1) \times (0, \infty)$ for which the discrete time approximation Y_t is p -stable with time step size Δ , when applied to the test equation (5.1).

By defining the random variable

$$G_{n+1}(\lambda\Delta, \alpha) = \left| \frac{Y_{n+1}}{Y_n} \right|,$$

which is called the transfer function of the approximation Y_t at time t_n , Platen and Shi [12] derive the following useful result which can determine the stability regions for given schemes by the following theorem.

Theorem 5.1. *A discrete time approximation is for given $\lambda < 0$, $\alpha \in [0, 1)$ and $p > 0$, p -stable if and only if*

$$E((G_{n+1}(\lambda\Delta, \alpha))^p) < 1.$$

In the spirit of Platen and Shi [12], stability regions for a range of schemes of SDEwMSs are discussed in this paper. Here we consider the test process $X_{r(t)} = \{X_{t,r(t)}, t \geq 0\}$ satisfies the linear SDEwMSs with multiplicative noise

$$dX_t = (1 - \frac{3}{2}\alpha(r(t)))\lambda(r(t))X_t dt + \sqrt{\alpha(r(t))|\lambda(r(t))|}X_t dW_t, \quad (5.3)$$

for every $t \geq 0$, where $r(t)$ is a Markov chain taking values in a finite state space $S = \{1, 2, \dots, N\}$, $r(0) = i_0 \in S$, $X_0 = X_{r(0)} \geq 0$ and $(\alpha(r(t)), \lambda(r(t))) \in \{(\alpha(i), \lambda(i)), \alpha(i) \in [0, 1], \lambda(i) < 0, i = 1, 2, \dots, N\}$. It is well known that the explicit solution of the test equation (5.3) is (see Mao and Yuan [11])

$$X_t = X_0 \exp\left\{\int_0^t (1 - \alpha(r(t)))\lambda(r(t))dt + \int_0^t \sqrt{\alpha(r(t))|\lambda(r(t))|}dW_t\right\}. \quad (5.4)$$

For the convenience of numerical comparison, we introduce the following stability criterion:

Definition 5.1. For $p > 0$, a process $Y_{r(t)} = \{Y_{t,r(t)}, t > 0\}$ is called state- p -stable if for each $i = 1, 2, \dots, N$, $Y_i = \{Y_{t,i}, t > 0\}$ satisfies

$$\lim_{t \rightarrow \infty} E(|Y_{t,i}|^p) = 0.$$

Definition 5.2. The state-stability region Γ_s is determined by those triplets $(\lambda\Delta, \alpha, p) \in (-\infty, 0) \times [0, 1) \times (0, \infty)$ for which the discrete time approximation $Y_{r(t)}$ is state- p -stable, when applied to the test equation (5.3), for each $i = 1, 2, \dots, N$, $(\alpha(i), \lambda(i)\Delta, p) \in \Gamma_s$ with time step size Δ .

Then we can obtain the following conclusion:

Theorem 5.2. The state-stability region Γ_s , which is generated by one algorithm applied to the test equation (5.3), is the same as the stability region Γ , which is generated by the same algorithm applied to the test equation (5.1).

Proof. $\Gamma \subseteq \Gamma_s$ is obvious. To prove $\Gamma_s \subseteq \Gamma$, suppose for all $i = 1, 2, \dots, N$, $(\alpha(i), \lambda(i)\Delta, p) \in \Gamma_s$. By Definition 5.2 we can see that $Y_{r(t)}$ is state- p -stable. And by Definition 5.1 we know for each $i = 1, 2, \dots, N$, $Y_i = \{Y_{t,i}, t > 0\} = \{Y_{t,(\alpha(i), \lambda(i)\Delta, p)}, t > 0\}$ is p -stable, so $(\alpha(i), \lambda(i)\Delta, p) \in \Gamma$, for all $i = 1, 2, \dots, N$. Immediately, we have $\Gamma_s \subseteq \Gamma$. \square

Remarks 5.1. By the conclusions derived in Platen and Shi [12], we can also see that the PCEM methods are more efficient than the EM method under these conditions in SDEwMSs case.

6 Numerical examples

In this section, we discuss two numerical examples to illustrate our theory established in the previous sections. Let us now consider several combinations of parameters θ and η in equation (2.2), the names of the methods listed below are similar to those used in Bruti-Liberati and Platen [3] and Platen and Shi [12].

- (1) $\theta = 0, \eta = 0$ (called EM scheme),
- (2) $\theta = \frac{1}{2}, \eta = \frac{1}{2}$ (called symmetric PCEM scheme),
- (3) $\theta = \frac{1}{2}, \eta = 0$ (called semi-drift-implicit PCEM scheme),
- (4) $\theta = 1, \eta = 0$ (called drift-implicit PCEM method),
- (5) $\theta = 0, \eta = \frac{1}{2}$ (called semi-diffusion-implicit PCEM scheme),
- (6) $\theta = 1, \eta = 1$ (called fully implicit PCEM scheme).

For a given problem, we will compare the simulated paths for these different degrees of implicitness. If these paths differ significantly from each other, then some numerical stability problem is likely to be present and one needs to make an effort in providing extra numerical stability for further research.

Example 6.1. Let $W(t)$ be a scalar Brownian motion. Let $r(t)$ be a right-continuous Markov chain taking values in $S = \{1, 2\}$ with generator

$$Q = \begin{bmatrix} -1 & 1 \\ q & -q \end{bmatrix}$$

And $W(t)$ and $r(t)$ are assumed to be independent. Consider an one-dimensional linear SDWMS

$$dy(t) = y(t)a(r(t))ds + y(t)b(r(t))dW(t) \quad (6.1)$$

for $t \geq 0$, where

$$\begin{cases} a(1) = 1, & b(1) = 2 \\ a(2) = 2, & b(2) = 1 \end{cases}$$

It is well known that equation (6.1) has an explicit solution

$$y(t) = y_0 \exp\left[\int_0^t a(r(s))ds + \int_0^t (r(s))dW(s) - \frac{1}{2} \int_0^t b^2(r(s))ds\right]. \quad (6.2)$$

For simulation reason, it is convenient to transform (6.2) into following recursion form with $y(t_0) = y_0$,

$$\begin{aligned} y(t_{k+1}) &\approx y(t_k) \exp[(t_{k+1} - t_k)a(r_{t_k}) + (W(t_{k+1}) - W(t_k))b(r_{t_k}) \\ &\quad - \frac{1}{2}(t_{k+1} - t_k)b^2(r_{t_k})]. \end{aligned} \quad (6.3)$$

Notice that $y(t_{k+1})$ in (6.3) is not the exact value of $y(t)$ at the division points t_{k+1} , because $r(s)$ is not necessarily constant on $[t_k, t_{k+1}]$. However, since

$$P\{r(t_{k+1}) = i | r(t_k) = i\} = 1 + q_{ii}(t_{k+1} - t_k) + o(t_{k+1} - t_k) \rightarrow 1$$

as $\Delta \downarrow 0$, for sufficiently small Δ , it is reasonable to use (6.3) as an approximation of the exact solution of $y(t)$.

Case 1. Let $q = 2$, $y_0 = 200$, $r_0 = 1$, $\Delta = 10^{-5}$, $k = 0, 1, \dots, 5 \times 10^6$, namely for the corresponding time $0 \leq t \leq 50$. Compute the one-step transition probability matrix

$$Q(\Delta) = \begin{bmatrix} 0.99999 & 0.00001 \\ 0.00002 & 0.99998 \end{bmatrix}$$

for the discrete Markov chain $r_{t_k} = r(k\Delta)$.

By applying the previously described procedure, the trajectory of the approximate solution $Y(t)$ with given stepsize Δ can be constructed. In this paper, we do not draw the figure of the simulating trajectory, instead, to carry out the numerical simulation clearly, we repeatedly simulate and compute $\sup_{t_k \in [0, 50]} (|Y_{(\theta_i, \eta_i)}(t_k) - y(t_k)|^2)$ ($i = 1, 2, 3, 4, 5, 6$) for 1000 times, then

calculate the sample mean $\widehat{E}(\sup_{t_k \in [0, 50]} |Y_{(\theta_i, \eta_i)}(t_k) - y(t_k)|^2)$ ($i = 1, 2, 3, 4, 5, 6$).

The results are listed in the following Table 1.

θ_i, η_i	$\widehat{E}(\sup_{t_k \in [0, 50]} Y_{(\theta_i, \eta_i)}(t_k) - y(t_k) ^2)$
0, 0	1.90674403017316e+30
$\frac{1}{2}, \frac{1}{2}$	7.84001334096008e+25
$\frac{1}{2}, 0$	1.90607879136196e+30
1, 0	1.90541425316912e+30
$0, \frac{1}{2}$	7.21187503884961e+25
1, 1	1.99576630102430e+30

Table 1. Estimation of the errors between the numerical solutions and exact solution

Case 2. Let $q = 1.5$, $y_0 = 200$, $r_0 = 1$, $\Delta = 10^{-5}$, $k = 0, 1, \dots, 5 \times 10^6$, namely for the corresponding time $0 \leq t \leq 50$. Compute the one-step transition probability matrix

$$Q(\Delta) = \begin{bmatrix} 0.99999 & 0.00001 \\ 0.000015 & 0.999985 \end{bmatrix}$$

for the discrete Markov chain $r_{t_k} = r(k\Delta)$.

To carry out the numerical simulation we repeatedly simulate and compute $\sup_{t_k \in [0, 50]} (|Y_{(\theta_i, \eta_i)}(t_k) - y(t_k)|^2)$ ($i = 1, 2, 3, 4, 5, 6$) for 1000 times, then

calculate the sample mean $\widehat{E}(\sup_{t_k \in [0,50]} |Y_{(\theta_i, \eta_i)}(t_k) - y(t_k)|^2)(i = 1, 2, 3, 4, 5, 6)$.

The results are listed in the following Table 2.

θ_i, η_i	$\widehat{E}(\sup_{t_k \in [0,50]} Y_{(\theta_i, \eta_i)}(t_k) - y(t_k) ^2)$
0, 0	3.33550923616514e+36
$\frac{1}{2}, \frac{1}{2}$	1.71565111220697e+33
$\frac{1}{2}, 0$	3.34748775539284e+36
1, 0	3.35948703848527e+36
$0, \frac{1}{2}$	3.51753180196352e+31
1, 1	3.23681904239271e+36

Table 2. Estimation of the errors between the numerical solutions and exact solution

Case 3. Let $q = 0.5$, $y_0 = 200$, $r_0 = 1$, $\Delta = 10^{-5}$, $k = 0, 1, \dots, 5 \times 10^6$, namely for the corresponding time $0 \leq t \leq 50$. Compute the one-step transition probability matrix

$$Q(\Delta) = \begin{bmatrix} 0.99999 & 0.00001 \\ 0.000005 & 0.999995 \end{bmatrix}$$

for the discrete Markov chain $r_{t_k} = r(k\Delta)$.

To carry out the numerical simulation we repeatedly simulate and compute $\sup_{t_k \in [0,50]} (|Y_{(\theta_i, \eta_i)}(t_k) - y(t_k)|^2)$ ($i = 1, 2, 3, 4, 5, 6$) for 1000 times, then

calculate the sample mean $\widehat{E}(\sup_{t_k \in [0,50]} |Y_{(\theta_i, \eta_i)}(t_k) - y(t_k)|^2)(i = 1, 2, 3, 4, 5, 6)$.

The results are listed in the following Table 3.

θ_i, η_i	$\widehat{E}(\sup_{t_k \in [0,50]} Y_{(\theta_i, \eta_i)}(t_k) - y(t_k) ^2)$
0, 0	1.44025299136110e+71
$\frac{1}{2}, \frac{1}{2}$	4.66862461497885e+66
$\frac{1}{2}, 0$	1.39047511677566e+71
1, 0	1.34154643977816e+71
$0, \frac{1}{2}$	5.41110836832358e+67
1, 1	1.42636805044311e+71

Table 3. Estimation of the errors between the numerical solutions and exact solution

This example has been studied in Yuan and Mao [18] in which Case 1, Case 2 and Case 3 represent three different exponential stability or instability situations respectively. However, since we can easily show that the equation (6.1) does not satisfy the conditions of equation (5.3), it will be not state- p -stable, the computer simulation results in Case 1, Case 2 and Case 3 illustrate this point in some extent. Nevertheless, when the parameters θ and η are selected reasonably, some PCEM methods are much more

efficient than the EM method in a certain extent.

Example 6.2. Let $r(t)$ be a right-continuous Markov chain taking values in $S = \{1, 2\}$ with generator

$$Q = \begin{bmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}$$

therefore, the one step transition probability from $r(t_k)$ to $r(t_{k+1})$ is $e^{Q\Delta}$.

Consider the same 1-dimensional linear SDWwMS

$$dy(t) = y(t)a(r(t))ds + y(t)b(r(t))dW(t) \quad (6.4)$$

on $t \geq 0$, this time let $a(r(t))$, $b(r(t))$ take values as follows

$$\begin{cases} a(1) = 0.15, & b(1) = 0.1 \\ a(2) = 0.05, & b(2) = 0.1 \end{cases}$$

Choose initial values $y_0 = 10$, $r_0 = 1$, T is fixed at 10. By applying the previously described procedure, the trajectory of the approximate solution $Y(t)$ with given stepsize Δ can be constructed.

To carry out the numerical simulation we successively choose the stepsize Δ as the following Table 4, and for each Δ , we repeatedly simulate and compute $\sup_{t_k \in [0, 10]} (|Y_{(\theta_i, \eta_i)}(t_k) - y(t_k)|^2)$, ($i = 1, 2, 3, 4, 5, 6$) for 1000 times, then

calculate the sample mean $\widehat{E}(\sup_{t_k \in [0, 10]} |Y_{(\theta_i, \eta_i)}(t_k) - y(t_k)|^2)$ ($i = 1, 2, 3, 4, 5, 6$).

The results are listed in the following Table 4.

$\widehat{E}(\sup_{t_k \in [0, 10]} Y_{(\theta_i, \eta_i)}(t_k) - y(t_k) ^2)$						
$\Delta \setminus \theta_i, \eta_i$	0, 0	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{2}, 0$	1, 0	$0, \frac{1}{2}$	1, 1
0.1	9702.4683	15.5672	1421.4360	8805.6057	7629.3505	7912.6245
0.01	376.9640	0.2299	265.3262	457.8189	150.4551	342.0570
0.001	24.0510	0.0017	23.6959	25.8235	1.1710	23.8747
0.0001	3.0465	2.16e-05	3.1024	3.1944	0.0158	3.0511
0.00001	0.1968	1.48e-07	0.1969	0.1971	9.73e-05	0.1968

Table 4. Estimation of the errors between the numerical and exact solutions

Clearly, we can easily show that the equation (6.4) satisfies the conditions of equation (5.3), so it will be state- p -stable, and the simulation results listed in Table 4 just illustrate the stability properties in a certain extent. On the one hand, the numerical method reveals that the numerical solution $Y(t)$

defined by the strong PCEM methods converge to the exact solution $y(t)$ in L^2 as step size $\Delta \downarrow 0$, and the order of convergence is one-half, i.e.

$$E(\sup_{0 \leq t \leq T} |Y(t) - y(t)|^2) \leq C\Delta + o(\Delta).$$

On the other hand, when the parameters selection are reasonable, the PCEM methods is much more efficient than the EM method, which strongly demonstrate our results.

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